# Interpolating sparse polynomials

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To the memory of Andreas Weber

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# Lagrangian interpolation

Let $f = \sum_{0 \le i \le d} a_i X^i$ be a polynomial of degree $d$. Let $y_0, \dots, y_d$ be pairwise distinct.

**Interpolation**: how to restore $f$ from values $f(y_0), \dots, f(y_d)$?

So, we always assume that $f$ is given by a black-box allowing to calculate $f$ at a given point.

$$f = \sum_{0 \le j \le d} f(y_j) \cdot \frac{(X - y_0) \cdots (X - y_{j-1})(X - y_{j+1}) \cdots (X - y_d)}{(y_j - y_0) \cdots (y_j - y_{j-1})(y_j - y_{j+1}) \cdots (y_j - y_d)}.$$

Proof. The values of two polynomials both of degrees $d$ in the left-hand and right-hand sides are equal at $d + 1$ points $y_0, \dots, y_d$, therefore, these two polynomials coincide.

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# Sparse polynomials

Polynomial $f = \sum_{1 \le i \le t} a_i X^{b_i}$ is $t$-**sparse**.

Informally: the number $t$ of monomials is much smaller than $\deg(f) = \max_{1 \le i \le t}\{b_i\}$.

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# Eigen-values and eigen-functions

$D$ is a linear operator on a space of functions (for example, complex or real).

Assume that for any eigen-value $\lambda$ of $D$ the eigen-space $E_\lambda := \{u : Du = \lambda u\}$ has dimension $\dim(E_\lambda) = 1$.

Assume also that there exists a constant $c$ (complex or real) such that $u(c) \neq 0$ for any eigen-value $\lambda$ and any eigen-function $0 \neq u \in E_\lambda$.

Pick an arbitrary $0 \neq u_0 \in E_\lambda$, then $E_\lambda = \{u = au_0 : a \in \mathbb{C} \text{ or } a \in \mathbb{R}\}$, and thereby, it suffices to have a value $u(c)$ to find $a = u(c)/u_0(c)$.

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# Interpolating sparse eigen-functions

Function $f$ is $t$**-sparse** (with respect to operator $D$) if $f = u_1 + \cdots + u_t$ where $u_1, \dots, u_t$ are eigen-functions of $D$ (with some eigen-values $\lambda_1, \dots, \lambda_t$, respectively).

Our goal is to interpolate $t$-sparse function knowing just $t$.

Assume that having a black-box for $f$ we have also a black-box for $Df$.

Involving the black-boxes the algorithm calculates $f(c), (Df)(c), (D^2 f)(c), \dots, (D^{2t} f)(c)$.

Note that $(D^i f) = \sum_{1 \le j \le t} \lambda_j^i u_j$, hence $(D^i f)(c) = \sum_{1 \le j \le t} \lambda_j^i u_j(c)$.

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# Generalized Wronskian

Consider the following $t \times t$ Wronskian (being a Hankel matrix)

$$W_t = \begin{pmatrix} f(c) & (D^1 f)(c) & \cdots & (D^{t-1} f)(c) \\ (D^1 f)(c) & (D^2 f)(c) & \cdots & (D^t f)(c) \\ \cdots & \cdots & \cdots & \cdots \\ (D^{t-1} f)(c) & (D^t f)(c) & \cdots & (D^{2t-2} f)(c) \end{pmatrix} =$$

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The first (and the third) matrices in the right-hand side are Vandermond (and its transposed) matrices, therefore $W_t$ is non-singular.
Consider (a unique) polynomial $g = X^t + \sum_{0 \le i < t} e_i X^i$ with the roots $\lambda_1, \dots, \lambda_t$. Then $W_{t+1} \cdot (e_0, \dots, e_{t-1}, 1)^T = 0$. Hence $rank(W_{t+1}) = t$ since $W_{t+1}$ contains $W_t$ as a submatrix in the upper left corner and $W_t$ is non-singular, thus $(e_0, \dots, e_{t-1}, 1)$ is the unique (normalized) solution of a homogeneous linear system $W_{t+1} E = 0$.

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# Interpolation algorithm for sparse eigen-functions

The algorithm (due to G.-Karpinski-Singer) finds

1. $(e_0, \ldots, e_{t-1}, 1)$ as a solution of a homogeneous linear system;
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# Applications

## Sparse polynomial interpolation in Pochhammer basis

Linear operator $(Df)(X) := X(f(X) - f(X-1))$ on the linear space of polynomials with the Pochhammer basis of the eigen-function of $D$ being $u_k = X(X-1)(X-2)\cdots(X-k+1)$, $k \geq 0$ with eigen-values $k$.

## Sparse Fourier decomposition

Linear operator $Df := f''$ acts as the second derivative on the space of continuous functions with the Fourier basis of eigen-functions $\sin(kX)$, $k \geq 1$ of $D$ with eigen-values $-k^2$.

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Linear operator $(Df)(X_1, \dots, X_n) := f(p_1 X_1, \dots, p_n X_n)$ where $p_1, \dots, p_n$ are distinct primes, acts on the space of polynomials with the basis of eigen-functions of $D$ being monomials $X_1^{s_1} \cdots X_n^{s_n}$ having distinct eigen-values $p_1^{s_1} \cdots p_n^{s_n}$.

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# Further developments

## Interpolation over finite fields

One can interpolate sparse polynomials over finite fields.

The latter algorithm fails over finite fields since eigen-values $p_1^{s_1} \cdots p_n^{s_n}$ can coincide for different monomials, therefore the dimensions of eigen-spaces can be greater than 1.

## Interpolation of sparse rational functions

Rational function $f$ is $(t_1, t_2)$-**sparse** if $f = f_1/f_2$ where polynomial $f_1$ (respectively, $f_2$) is $t_1$-sparse (respectively, $t_2$-sparse).

Note that the irreducible representation of a rational function is not necessary sparse: $1 + X + X^2 + \cdots + X^d = (X^{d+1} - 1)/(X - 1)$.

For given $f, t$ the interpolation algorithm produces all the $(t_1, t_2)$-sparse representations of $f$ with $t_1 + t_2 \leq t$.

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