Decoupling Multivariate Fractions

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Symbiont meeting

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Summary

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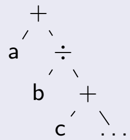
Purpose of the talk

Find a nice/compact form of a fraction

- what does nice/compact mean ?
- $a + \dfrac{b}{c + \frac{d}{e+f}}$ or $\dfrac{ace + acf + ad + be + bf}{ce + cf + d}$
- $cx - \dfrac{V_1 x}{k_1 + x} - \dfrac{V_2 x}{k_2 + x}$ or
  $\dfrac{ck_1 k_2 x + ck_1 x^2 + ck_2 x^2 + cx^3 - k_2 V_1 x - V_1 x^2 - k_1 V_2 x - V_2^2}{(k_1 + x)(k_2 + x)}$

Fractions representations

- “expanded” form: numerator / denominator
- tree/dag:



$\to$ suitable for nested fractions

Why is a compact form interesting ?

A compact form:

- is easier to read and understand for a human
- takes less memory
- may yield a better interval evaluation

Fractions are underrated ?

- many computer algebra algorithms:
  - operate on polynomials (taking numerators, . . .)
  - output polynomials
  - $\rightarrow$ Gröbner basis, triangular sets, . . .
- Multivariate partial fraction decomposition ?
  - Stoutemeyr 2009, “Multivariate partial fraction decomposition”
  - Leinartas 1978 “Factorization of rational functions of several variables into partial fractions“
  - Both approaches decompose a fraction as a sum of fractions, but no nested decomposition is recovered
- finding nested/compact forms for fractions seem quite difficult ...

Summary

Demo in Maple

Summary

Decoupling of a multivariate fraction

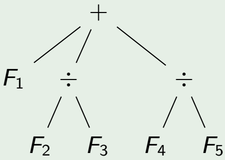
Definition

- let F a fraction of $\mathbb{K}(X)$, where X is a set of variables,
- let $X_1, \ldots, X_p$ a partition of the set X,
- F can be decoupled w.r.t. to the partition if F can be written as $F = A(F_1, F_2, \ldots, F_p)$ where :
  - $A(a_1, \ldots, a_p)$ is a tree with nodes $+$, $\times$ and $\div$,
  - each a_i appears at most once (at a leaf),
  - each F_i is a fraction in X_i.

Example

$X = \{c, V_1, k_1, V_2, k_2\}$, $\mathbb{K} = \mathbb{Q}(x)$, $F = cx - \frac{V_1 x}{k_1 + x} - \frac{V_2 x}{k_2 + x}$ can decoupled w.r.t. the partition $\{c\}, \{V_1\}, \{k_1\}, \{V_2\}, \{k_2\}$:

$F =$



with

- $F_1 = cx$
- $F_2 = -V_1 x$
- $F_3 = k_1 + x$
- $F_4 = -V_2 x$
- $F_5 = k_2 + x$.

Decoupling of a multivariate fraction

Properties

- for each fraction F, there exists a (unique) finest partition for decoupling F
- there is no uniqueness for the tree A and the F_i's

Interest

The more decoupled the fraction is:

- the shorter/more compact/nicer the fraction should look
- the better the interval evaluation should be

Example: better "looking" equations

(Example 4 of the paper, self-regulated gene)

$$H'(t) = -G'(t) = \frac{a((fM(t) - V_p)P(t) + fk_pM(t))G(t)}{(aG(t) + aP(t) + b)(k_p + P(t))},$$
$$M'(t) = \frac{(eG(t) - V_m)M(t) + ek_mG(t)}{k_m + M(t)},$$
$$P'(t) = \frac{((fM(t) - V_p)P(t) + fk_pM(t))(aP(t) + b)}{(aG(t) + aP(t) + b)(k_p + P(t))}.$$

becomes

$$H'(t) = -G'(t) = \frac{fM(t) - \dfrac{V_pP(t)}{k_p + P(t)}}{1 + \dfrac{\frac{b}{a} + P(t)}{G(t)}}$$

$$M'(t) = -\frac{V_m}{\dfrac{k_m}{M(t)} + 1} + eG(t) \qquad P'(t) = \frac{fM(t) - \dfrac{V_pP(t)}{k_pP(t)}}{1 + \dfrac{G(t)}{P(t)\left(1 + \frac{b}{aP(t)}\right)}}.$$

Example: better interval evaluation

(Example 5 of the paper)

$$F = \frac{a_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + a_3}}}{c_0 + \frac{c_1}{d_1 + \frac{c_2}{d_2 + c_3}}} + \frac{e_0 + \frac{e_1}{f_1 + \frac{e_2}{f_2 + e_3}}}{g_0 + \frac{g_1}{h_1 + \frac{g_2}{h_2 + g_3}}}$$

- developing F yields P/Q with P and Q of degree 10, and more than 200 monomials
- our algorithm retrieves F (up to some signs) from P/Q
- if each variables lies in the interval $[1.0, 5.0]$,
  - the fraction P/Q yields the interval $[0.140 \times 10^{-5}, 0.284 \times 10^{7}]$
  - the decoupled form yields $[0.237, 16.8]$.

Note: this is of course an extreme case scenario

Summary

A simple recursive algorithm

$decouple(F, X)$, where $F \in \mathbb{K}(X)$

if F cannot be decoupled for any partition of X then
    return F
else
    find a partition (Y, Z) of X such that F can written as $A(G(Y), H(Z))$
    call $decouple(G, Y)$ and $decouple(H, Z)$ and build the result

How to find the partition (Y, Z) ?

Central theorem

If a fraction $F \in \mathbb{K}(X)$ can be decoupled w.r.t. a partition (Y, Z) of X, then F can be written as one the following form:

- C1 $G(Y) + H(Z)$
- C2 $c + G(Y)H(Z)$
- C3 $c + \dfrac{1}{G(Y) + H(Z)}$
- C4 $c + \dfrac{d}{1 + G(Y)H(Z)}$ with

where c and d are in (almost!) $\mathbb{K}$, and $d \neq 0$.

- the four cases are exclusive
- cases C1 and C2 are sufficient if F is a polynomial
- case C1 amounts to a simple graph traversal
- cases C2,C3,C4 are more difficult: the constants c and d are computed if they exist, but how ?

Case C2 with differential algebra !

Assume

- $f(x,y) = c + g(x)h(y)$, with c constant
- x and y are scalar
- f, g and h are smooth enough
- f_{xy} is not the zero function

Then $c = f - \dfrac{f_x f_y}{f_{xy}}$

With DifferentialAlgebra (Maple package by François Boulier)

Eliminate g and h in the following system

$$f = c + gh \tag{1}$$

$$c_x = c_y = 0 \tag{2}$$

$$g_y = 0 \tag{3}$$

$$h_x = 0 \tag{4}$$

Case C4 may take you outside the base field $\mathbb{K}$

Over $\mathbb{K} = \mathbb{Q}(a)$, the fraction $p = \frac{xy+a}{x+y}$ in $\mathbb{K}(x, y)$ can be decoupled w.r.t. $\{x\}, \{y\}$:

$$-\sqrt{a} + \frac{2\sqrt{a}}{1 + \left(1 - \frac{2\sqrt{a}}{x+\sqrt{a}}\right)\left(-1 + \frac{2\sqrt{a}}{y+\sqrt{a}}\right)}$$

- if you plug $a = 2$, then $\mathbb{K} = \mathbb{Q}$ and you get a $\sqrt{2}$
- if you plug $a = -1$, then $\mathbb{K} = \mathbb{Q}$ and you get a complex I
- no escape is possible … reason: the constants c and d (case C4) are solutions of a second order equation
- you sometimes need to take radicals to decouple a fraction (this was quite surprising)

Conclusion

- the code requires some heuristics to avoid too many differentiations (many evaluations tricks are used)
- the complexity is polynomial in the number of operations on fractions