Quasi-Linear Differential Equations

Two Second Order Equations

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Objectives

Existence, (Non)Uniqueness and Regularity
of Solutions
of an Initial Value Problem

$$u(\overline{x}) = \overline{u}, \quad u'(\overline{x}) = \overline{u}'$$

for a Second Order ODE of Type

$$g(x)u'' = f(x, u, u')$$

with $\overline{x}$ a zero of $g(x)$.

Two Cases:
Initial condition $\overline{x}$ is a simple or a double zero of $g(x)$

Why relevant for Symbiont?

Connection between quasi-linear systems
$$A(\boldsymbol{u})\boldsymbol{u}' = F(\boldsymbol{u})$$
and singular pertubation problems
$$\boldsymbol{u}' = f(\boldsymbol{u}, \boldsymbol{v}, \epsilon),\ \epsilon\boldsymbol{v}' = g(\boldsymbol{u}, \boldsymbol{v}, \epsilon)$$

An Analytic View

Theorem (Liang 2009)

Let $g(x) = x$ and set $\gamma = f_{u'}(0, \overline{u}, \overline{u}')$, $k = \lceil \gamma \rceil$

- $\gamma < 0 \Rightarrow$ *unique solution* $u \in C^\infty([0, \delta])$ *for some* $\delta > 0$.
- $\gamma > 0 \Rightarrow \exists \overline{u}^{(2)}, \dots, \overline{u}^{(k)} \in \mathbb{R}$ *(uniquely determined by* $\overline{u}, \overline{u}', f$*): for every solution*

  $$\lim_{x \to 0+} \frac{1}{x^\gamma} \Big(u'(x) - \sum_{j=1}^{k} \frac{\overline{u}^{(k)} x^{j-1}}{(j-1)!} \Big)$$

  exists, each solution uniquely determined by limit; one solution smooth, all others in $C^k \setminus C^{k+1}$.
- $\gamma \in \mathbb{N} \Rightarrow$ *additional logarithmic term and either all solutions are smooth or all in* $C^k \setminus C^{k+1}$ *("resonance").*

Theorem (Brunovsky, Cerny, Winkler 2013)

$$x^2 u'' = axu' + bu - c(u' - 1)^2, \quad x \in (0, x_0)$$

with $a, b \in \mathbb{R}$, $c > 0$ *and initial condition* $u(0) = 0$.

- $a + b < 0 \Rightarrow$ *no continuous solutions exist*
- $a + b > 0 \Rightarrow$ *infinitely many continuous solutions exist;*
  unique solution u *corresp. to the choice* $x_0 = \infty$ *with* $0 \le u(x) \le x$ *for all* $x > 0$
  and u *is strictly increasing and concave.*

The Geometry of ODEs

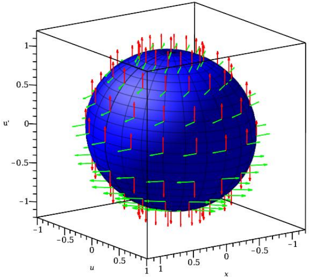
1. **Jet bundle** $\mathcal{J}_q$ the set of all equivalence classes $[\phi]_x^{(q)}$ of smooth functions $\phi : \mathbb{R} \to \mathbb{R}$ with the same Taylor expansion at x up to order q
2. Coordinates $(x, u, u', \dots, u^{(q)})$
3. Natural **projection** $\pi_r^q : \mathcal{J}_q \to \mathcal{J}_r$ for $q > r$ and $\pi^q : \mathcal{J} \to \mathbb{R}$.
4. **The contact distribution** $\mathcal{C}^{(q)}$: It is spanned by the two vector fields
$$C^{(q)} = \partial_x + u^{(1)}\partial_u + \cdots + u^{(q)}\partial_{u^{(q-1)}} \quad (\textit{transversal})$$
$$\text{and} \quad C_q = \partial_{u^{(q)}} \quad (\textit{vertical})$$
5. Identify a scalar qth-order **differential equation** $F = 0$ as a submanifold $\mathcal{R}_q \subseteq \mathcal{J}_q$
6. **Vessiot space** $\mathcal{V}_\rho[\mathcal{R}_q] = T_\rho\mathcal{R}_q \cap \mathcal{C}^{(q)}|_\rho$ at $\rho \in \mathcal{R}_q$.
Computation: Every $X \in \mathcal{V}_\rho[\mathcal{R}_q]$ has a representation as
$$X = aC^{(q)}|_\rho + bC_q|_\rho$$
where a, b are the solutions of
$$C^{(q)}(F)(\rho)a + C_q(F)(\rho)b = 0$$

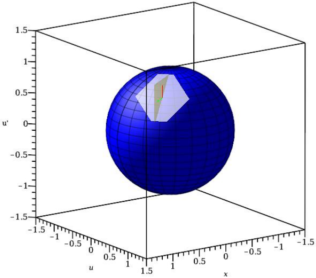
Example Sphere

Consider the equation

$$F = u'^2 + u^2 + x^2 - 1 = 0$$

in $\mathcal{J}_1$ with contact disrtibution $C^{(1)} = \partial_x + u'\partial_u$ and $C_1 = \partial_{u'}$.



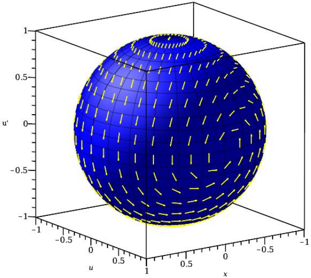


Example Sphere

Vessiot distribution:

$$aC^{(2)}(F) + bC_2(F) = a(2x + 2uu') + b2u' = 0$$

$$\rightsquigarrow X = u'\partial_x + u'^2\partial_u - (x + uu')\partial_{u'}$$

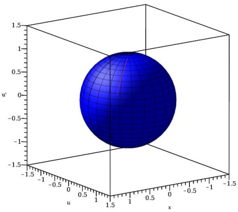


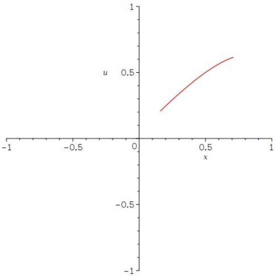
Solutions

1. Identify a function $\phi : \mathbb{R} \to \mathbb{R}$ with its **section** $\sigma : \mathbb{R} \to \mathbb{R}^2$ where $\sigma(x) = (x, \phi(x))$.
2. **Prolonged section**

$$j_q\sigma_\phi : \mathbb{R} \to \mathcal{J}_q, \quad x \mapsto (x, \phi(x), \phi'(x), \dots, \phi^{(q)}(x)).$$

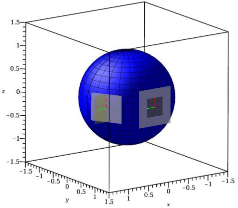
3. A function ϕ is a **strong solution** of $\mathcal{R}_q$ if $j_q\sigma_\phi \subseteq \mathcal{R}_q$
4. A **generalised solution** is a one dimensional integral manifold $\mathcal{N} \subseteq \mathcal{R}_q$ of $\mathcal{V}[\mathcal{R}_q]$.
5. A generalised solution is **proper** if there does not exist x such that $\mathcal{N} \subseteq (\pi^q)^{-1}(x)$.
6. The projection $\pi^q_0(\mathcal{N}) \subset \mathcal{J}_0$ of a proper generalised solution is called a **geometric solution**.





Singularities

Two Types of Singularities:

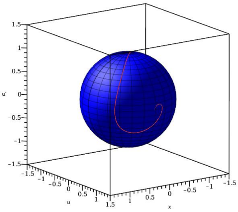


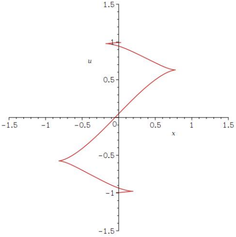
Definition

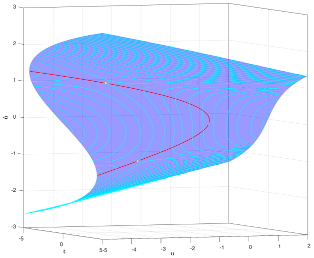
Let ρ be a smooth point of $\mathcal{R}_q$.

1. If $\dim(\mathcal{V}_\rho[\mathcal{R}_q]) > 1$, then ρ is called an irregular singularity.
2. If $\dim(\mathcal{V}_\rho[\mathcal{R}_q]) = 1$ and $\mathcal{V}_\rho[\mathcal{R}_q]$ is vertical, then ρ is called a regular singularity.
3. Otherwise ρ is called regular.

Examples Singularities and Solutions





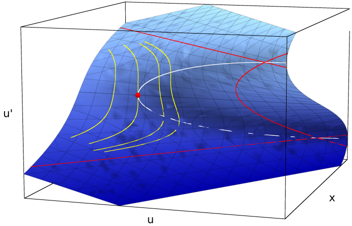


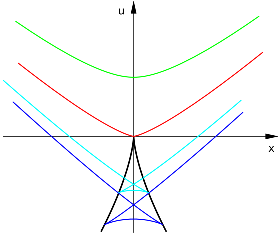
No Irregular Singularities

Theorem

Let $\mathcal{R}_q \subset \mathcal{J}_q$ be a scalar ordinary differential equation such that at every $\rho \in \mathcal{R}_q$ the Vessiot space $V_\rho[\mathcal{R}_q]$ is one-dimensional.

1. *If ρ is regular, then $\exists_1$ strong (two sided) solution σ with $\rho \in im(j_q\sigma)$.*
2. *If ρ is regular singular, then either two strong one sided solutions σ_1, σ_2 with $\sigma_i \in imj_q\sigma_i$ exist or only one strong two-sided solution exists whose $(q+1)th$ derivative blows up at $x = \pi^q(\rho)$.*





Prolongation

Recall: $\mathcal{R}_q$ is defined by the zero set of the function $F : \mathcal{J}_q \to \mathbb{R}$.
The first prolongation $\mathcal{R}_{q+1} \subseteq \mathcal{J}_{q+1}$ is the zero set defined by F and the function

$$D_x F = C^{(q)}(F) + C_q(F)u^{(q+1)} : \mathcal{J}_{q+1} \to \mathbb{R}$$

Proposition

Let $\mathcal{F}_{q+1} = (\pi_q^{q+1})(\rho_q) \cap \mathcal{R}_{q+1}$ be the fibre of a point $\rho_q \in \mathcal{R}_q$ in the first prolongation $\mathcal{R}_{q+1}$.

1. *If ρ_q is regular, then $\mathcal{F}_{q+1} \neq \emptyset$ and consists of regular points of $\mathcal{R}_{q+1}$.*
2. *If ρ_q is regular singular, then $\mathcal{F}_{q+1} = \emptyset$.*
3. *If ρ_q is irregular singular, then $\mathcal{F}_{q+1} \neq \emptyset$ and consists of singular points of $\mathcal{R}_{q+1}$.*

Quasi-Linear Equations

We consider now quasi-linear equations type

$$g(x, \boldsymbol{u}_{q-1})u^{(q)} = f(x, \boldsymbol{u}_{q-1}).$$

Equation determining the Vessiot space at a point $\rho = (x, \overline{\boldsymbol{u}}_{q-1}) \in \mathcal{R}_q$ becomes

$$[C^{(q)}(g)(\rho)\overline{u}^{(q)} - C^{(q)}(f)(\rho)]a + g(\rho)b = 0$$

$\Rightarrow$ a point ρ is singular if $g(\rho) = 0$ (independent of $\overline{u}^{(q)}$) and it is irregular singular if and only if in addition

$$C^{(q)}(g)(\rho)\overline{u}^{(q)} - C^{(q)}(f)(\rho) = 0.$$

The Vessiot space is

$$X = g(x, \boldsymbol{u}_{q-1})C^{(q)} - [C^{(q)}(g(x, \boldsymbol{u}_{q-1}))u^{(q)} - C^{(q)}(f(x, \boldsymbol{u}_{q-1}))]C_q.$$

Key Property

Key Property of Quasi-linear Differential Equations:

X is π^q_{q-1}-projectable to

$$Y = g(x, \boldsymbol{u}_{q-1})C^{(q-1)} + f(x, \boldsymbol{u}_{q-1})C_{q-1}.$$

Assumption: Y is continueable to all of $\mathcal{J}_{q-1}$.

Definition

1. A point $\widetilde{\rho}_{q-1} \in \mathcal{J}_{q-1}$ is called **impasse point** for $\mathcal{R}_q \subseteq \mathcal{J}_q \iff Y$ is not transversal to $\pi^{(q-1)}$ at $\widetilde{\rho}_{q-1}$ (∂_x-component vanishes)
2. An impasse point $\widetilde{\rho}_{q-1} \in \mathcal{J}_{q-1}$ is called **proper** $\iff Y$ vanishes at $\widetilde{\rho}_{q-1}$. Otherwise it is called improper.

The local analysis of proper impasse points reduces to the study of the stationary points of the vector field Y.

Impasse Points of Quasi-Linear Equations

Definition

1. A one-dimensional invariant manifold of Y is called a **weak generalised solution**.
2. The π_0^{q-1}-projection of a weak generalised solution is called a **weak geometric solution**.

No guarantee that qth derivative exists $\rightsquigarrow$ generally not related to a classical solution

First Type of Quasi-Linear Equations

Problem 1:

For a quasi-linear differential equation of type

$$g(x)u'' = f(x, u, u')$$

we consider the singular initial value problem $\rho_1 = (\overline{x}, \overline{u}_0, \overline{u}_1)$ *where* $f(\rho_1) = 0$ *and* $\overline{x}$ *is a* **simple zero** *of* $g(x)$.

Prolonged Equations and Vessiot Distribution

For $q > 2$ compute recursively the prolonged equations $\mathcal{R}_q$ defined by $F_2, \ldots, F_{q-1}$ and

$$F_q(x, \boldsymbol{u}_{(q)}) = g(x)u^{(q)} + [(q-2)g'(x) - f_{u'}(x, u, u')]u^{(q-1)} - h_q(x, \boldsymbol{u}_{(q-2)})$$

where

$$\begin{aligned} h_3(x, \boldsymbol{u}_{(1)}) &= C^{(1)}\big(f(x, u, u')\big) \\ h_q(x, \boldsymbol{u}_{(q-2)}) &= C^{(q-2)}\Big(h_{q-1}(x, \boldsymbol{u}_{(q-3)}) - [(q-3)g'(x) - f_{u'}(x, u, u')]u^{(q-2)}\Big) \end{aligned}$$

On the prolonged equations $\mathcal{R}_q$ the Vessiot distribution is given by

$$X^{(q)} = g(x)C^{(q)} + \big([(q-1)g'(x) - f_{u'}(x, u, u')]u^{(q)} - h_{q+1}(x, \boldsymbol{u}_{q-1})\big)\, C_q$$

Singularities: $\rho_q = (\overline{x}, \overline{u}, \overline{u}', \ldots, \overline{u}^{(q)}) \in \mathcal{R}_q$

1. regular singularity $\iff$ $g(\overline{x}) = 0$
2. irregular singularity $\iff$ $g(\overline{x}) = 0$ and $[g'(\overline{x}) - f_{u'}(\rho_1)]\overline{u}^{(q)} = h_{q+1}(\rho_{q-1})$

where $\pi^q_i(\rho_q) = \rho_i$ for $i = q-1$ and $i = 1$.

The Dynamical System

The Projected Vessiot distribution is

$$Y^{(q-1)} = g(x)C^{(q-1)} + \left(h_q(x, \boldsymbol{u}_{(q-2)}) - [(q-2)g'(x) - f_{u'}(x, \boldsymbol{u}_{(1)})]u^{(q-1)}\right)C_{q-1}$$

For our singular initial values $\rho_1 = (\overline{x}, \overline{u}, \overline{u}')$ all points

$$\rho_q = (\overline{x}, \overline{\boldsymbol{u}}_q) \in \mathcal{R}_q \cap (\pi^q_1)^{-1}(\rho_1)$$

are singular too and the projection $\rho_{q-1} = \pi^q_{q-1}(\rho_q) \in \mathcal{R}_{q-1}$ is a proper impasse point of $\mathcal{R}_q$

$$\rightsquigarrow \text{ analyse the local phase portrait of } Y^{(q-1)} \text{ at } \rho_{q-1}$$

The Jacobian at the Singularity

We set $\delta = g'(\overline{x})$ and $\gamma = f_{u'}(\rho_1)$ and assume wlog that $\delta > 0$.

At $\rho_{q-1} = (\overline{x}, \overline{u}_0, \ldots, \overline{u}_{q-1})$ the Jacobian of $Y^{(q-1)}$ is

$$J^{(q-1)} = \begin{pmatrix} \delta & 0 & \cdots & & 0 \\ \delta\overline{u}_1 & 0 & \cdots & & 0 \\ \vdots & \vdots & & & \vdots \\ \delta\overline{u}_{q-1} & 0 & \cdots & & 0 \\ a_0 & \cdots & a_{q-1} & & \gamma - (q-2)\delta \end{pmatrix}$$

with eigenvalues δ, 0 ((q-1) times) and $\gamma - (q-2)\delta$.

$\mathcal{R}_{q-1}$ is a 3-dimensional manifold $\rightsquigarrow$

δ, 0 and $\gamma - (q-2)\delta$
are the relevant eigenvalues

Resonance

Definition

We say that the initial value problem at ρ_1 has a **resonance** at order $k \in \mathbb{N}$, if

$$k\delta = \gamma.$$

In this case at any point $\rho_k \in (\pi_1^k)^{-1}(\rho_1)$ above ρ_1 we call $A_k = h_{k+2}(\rho_k)$ the **resonance parameter**. The resonance is **critical** at ρ_k if $A_k \neq 0$ and **smooth** if $A_k = 0$.

Proposition

Let $\rho_q \in \mathcal{R}_q$ be an irregular singularity and let $\mathcal{F}_{q+1} = (\pi_q^{(q+1)})^{-1}(\rho_q)$ be the fibre.

1. *Then $\mathcal{F}_{q+1} \subset \mathcal{R}_{q+1}$*
2. *If ρ_1 is not in resonance at order q, then $\mathcal{F}_{q+1}$ contains exactly one irregular singularity.*
3. *If ρ_1 is in $\left\{ \begin{array}{c} \text{critical} \\ \text{smooth} \end{array} \right\}$ resonance at order q, then $\mathcal{F}_{q+1}$ consists entirely of $\left\{ \begin{array}{c} \text{regular} \\ \text{irregular} \end{array} \right\}$ singularity.*

Eigenvectors

Eigenvectors of $J^{(q-1)}$:

Without resonance:

- $\boldsymbol{e}_{\gamma-(q-2)\delta} = (0, \ldots, 0, 1)^T$ tangential to the fibre
- $\boldsymbol{e}_\delta = (1, \overline{u}', \ldots, \overline{u}^{(q-1)}, \frac{-h_{q+1}(\rho_{q-1})}{\gamma-(q-1)\delta})^T$ transversal to π^{q-1}
- $\boldsymbol{e}_0$ is tangential to the curve of proper impasse points

Resonance at order $q-1$: (Generalised) Eigenspace generated by

$$\boldsymbol{e}_1 = (0, \ldots, 0, 1)^T \quad \textit{and} \quad \boldsymbol{e}_2 = (1, \overline{u}', \ldots, \overline{u}^{(q-1)}, 0)^T$$

- Resonance smooth: $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are proper eigenvectors
- Resonance critical: $\boldsymbol{e}_1$ proper and $\boldsymbol{e}_2$ generalised eigenvector

The Case Without Resonance

Theorem

Assume that at no order a resonance appears.

1. *If $\delta\gamma < 0$, then the corresponding initial value problem possesses a unique two-sided smooth solution and no additional one-sided solutions.*
2. *If $\delta\gamma > 0$, then there exists a one-parameter family of two-sided solutions. One of these solutions is smooth; the other ones are in $\mathcal{C}^k \setminus \mathcal{C}^{k+1}$ with $k = \lceil \gamma/\delta \rceil$. All of these solutions possess the same Taylor polynomial $\sum_{i=0}^{k} \frac{\overline{u}_i}{i!}(x - \overline{x})^i$ of degree k around $\overline{x}$.*

The Case of $\gamma = 0$

Theorem

Assume that $\gamma = 0$. Then there exists a unique smooth two-sided solution (and possibly further one-sided solutions).

The Case of Resonance

Theorem

Assume that a resonance occurs at the order $k > 0$. There exists a one-parameter family of two-sided solutions all possessing the same Taylor polynomial $\sum_{i=0}^{k} \frac{\overline{u}_i}{i!}(x - \overline{x})^i$ of degree k around $\overline{x}$. In the case of a smooth resonance, all of these solutions are smooth and each is uniquely determined by the value of its $(k+1)$st derivative in $\overline{x}$. In the case of a critical resonance, all solutions live in $\mathcal{C}^k \setminus \mathcal{C}^{k+1}$.

Brunovsky

Problem 2:

For the quasi-linear differential equation

$$x^2 u'' = axu' + bu - c(u' - 1)^2$$

we consider the singular initial value problem $\rho_1 = (\overline{x}, \overline{u}_0, \overline{u}_1) = (0, 0, 1)$ *with are real parameters* $a, b, c \in \mathbb{R}$.

Joint work with Peter Szmolyan

The Case b, c are nonzero and $a \neq -b$

The projected Vessiot distribution is

$$Y = x^2 \partial_x + x^2 u' \partial_u + (axu' + bu - c(v-1)^2)\partial_{u'}.$$

The set of stationary points is given by the parabola

$$u = \frac{c}{b}(u' - 1)^2$$

and the Jacobian has at ρ_1 triple eigenvalue zero.

$\Rightarrow$ desingularise the stationary point ρ_1 with a blow-up

Consider the transformed system

$$(b(ay_1y_3 - cy_1^2) + y_2)\partial_{y_1} + (by_3^2(b^2y_1 + a + b))\partial_{y_2} + by_3^2\partial_{y_3}$$

with Jacobian

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

at the origin.

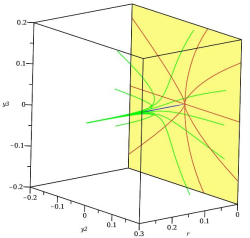
The Case b, c are nonzero and $a \neq -b$

Blow-up in positive and negative y_1 direction:

Stationary points:
$s^1 = (0,0,0)$, $s^2 = (0, bc, 0)$ (later)

$$J(s^1) = \begin{pmatrix} -bc/2 & 0 & 0 \\ 0 & 2bc & 0 \\ 0 & 0 & 3bc/2 \end{pmatrix}$$

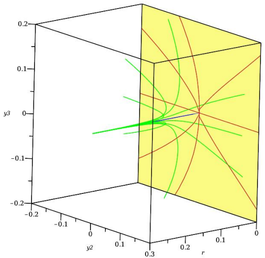
s^1 repelling/attracting node
y_1 direction attracting/repelling



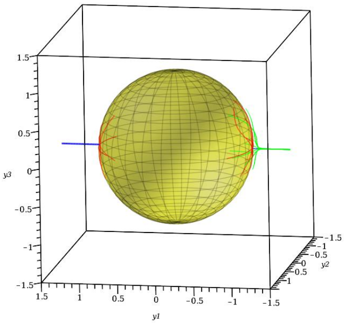
Stationary points:
$s^3 = (0, bc, 0)$, $s^4 = (0,0,0)$ (later)

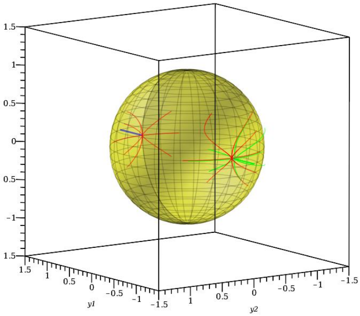
$$J(s^4) = \begin{pmatrix} bc/2 & 0 & 0 \\ 0 & -2bc & 0 \\ 0 & 0 & -3bc/2 \end{pmatrix}$$

s^4 attracting/repelling node
y_1 direction repelling/attracting



The Case b, c are nonzero and $a \neq -b$





The Case b, c are nonzero and $a \neq -b$

Blow up in positive y_2 direction (negative analogously):

Stationary points

$$s^2 = (\alpha, 0, 0) \text{ and } s^3 = (-\alpha, 0, 0) \text{ with } \alpha = \sqrt{\tfrac{1}{bc}}$$

with Jacobians

$$J(s^2) = \begin{pmatrix} -2bc\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J(s^3) = \begin{pmatrix} 2bc\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Lemma

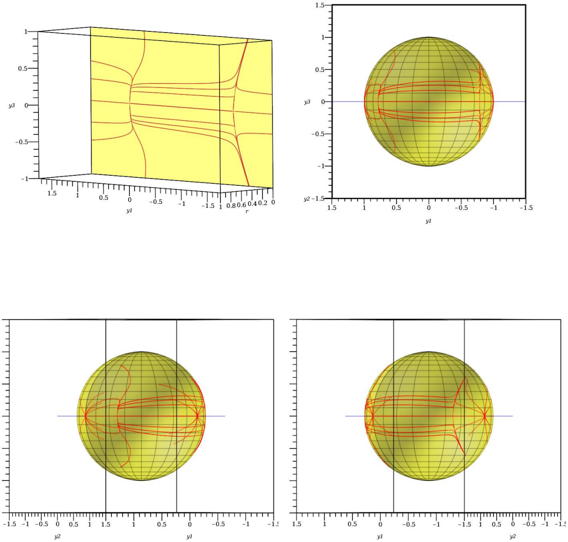
1. *Let $a + b \neq 0$ and $bc < 0$. Then in the chart for the positive y_2-direction there are no real stationary points on the plane $\{y_2 = 0\}$.*
2. *Let $a + b \neq 0$ and $bc > 0$. Then in the chart for the positive y_2-direction we find on the plane $\{y_2 = 0\}$ the two stationary points*

   $$s^2 = (\alpha, 0, 0) \quad \text{and} \quad s^3 = (-\alpha, 0, 0).$$

   On the plane $\{y_2 = 0\}$ they show the following dynamical behaviour:

   2.1 *For $b(a + b) > 0$ the point s^2 is an attracting node and s^3 is a saddle.*
   2.2 *For $b(a + b) < 0$ the point s^2 is a saddle and s^3 is a repelling node.*

The Case b, c are nonzero and $a \neq -b$



The Case b, c are nonzero and $a \neq -b$

At s^2 and s^3 the Jacobians have double eigenvalue zero

$\Rightarrow$ Compute center manifolds and their dynamics

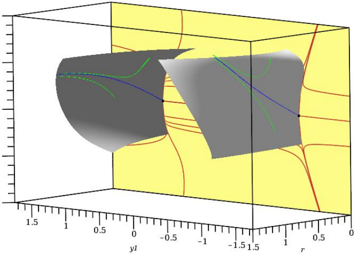
Lemma

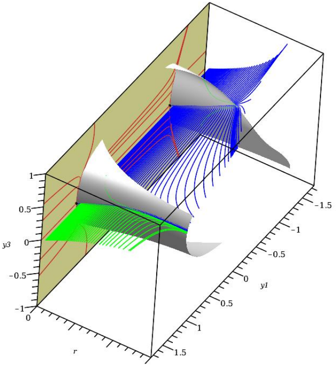
On the center manifold $\mathcal{C}(s^2)/\mathcal{C}(s^3)$ the point s^2/s^3 is a saddle point. On the halfspace $\{y_2 \geq 0\}$ we have the following dynamics on $\mathcal{C}(s^2)/\mathcal{C}(s^3)$:

1. *$a+b>0$, $b>0$: attracting directions $\pm(0,0,1)^{tr}$ and repelling direction $(0,a+b,1)^{tr}$.*
2. *$a+b>0$, $b<0$: repelling directions $\pm(0,0,1)^{tr}$ and attracting direction $(0,a+b,1)^{tr}$.*
3. *$a+b<0$, $b>0$: repelling directions $\pm(0,0,1)^{tr}$ and attracting direction $-(0,a+b,1)^{tr}$.*
4. *$a+b<0$, $b<0$: attracting directions $\pm(0,0,1)^{tr}$ and repelling direction $-(0,a+b,1)^{tr}$.*

1. *$a+b>0$, $b>0$: attracting directions $\pm(0,0,1)^{tr}$ and repelling direction $(0,a+b,1)^{tr}$.*
2. *$a+b>0$, $b<0$: repelling directions $\pm(0,0,1)^{tr}$ and attracting direction $(0,a+b,1)^{tr}$.*
3. *$a+b<0$, $b>0$: repelling directions $\pm(0,0,1)^{tr}$ and attracting direction $-(0,a+b,1)^{tr}$.*
4. *$a+b<0$, $b<0$: attracting directions $\pm(0,0,1)^{tr}$ and repelling direction $-(0,a+b,1)^{tr}$.*

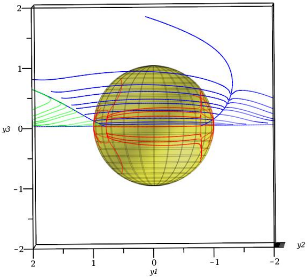
The Case b, c are nonzero and $a \neq -b$

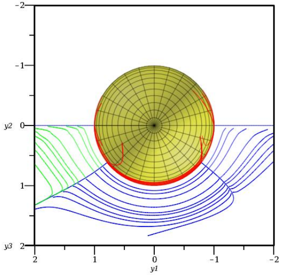




The Case b, c are nonzero and $a \neq -b$

Putting it together





$\Rightarrow$ the stationary point is a saddle node