On Denef and Lipshitz (1984) Theorem 3.1

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Tropical Differential Algebraic Geometry was founded by Dima (2015). It is an analogue of Tropical Algebra for systems of (polynomial) differential equations.

It is strongly related to existence problems of Formal Power Series (FPS) solutions of such systems.

A major, difficult, related paper is that (DL84) of Denef and Lipshitz (1984) in the Mathematische Annalen.

Existence Problems of FPS Solutions

Let Σ be a polynomial differential system.

	solution in $\mathbb{C}[[x]]$ (centered at the origin)	solution in $\mathbb{C}[[x-\alpha]]$ (unspecified α)
ODE	Decidable by [DL84, Thm 3.1]	Decidable (differential elimination theory)
PDE	Undecidable by [DL84, Thm 4.11]	Decidable (differential elimination theory)

Even in the [DL84, Thm 3.1] case:

1. the existence of nonzero solutions undecidable [Singer, 1978]
2. the problem is tricky: $P = x\,y\,\ddot{y} + \dot{y} + y^2 + 1$ has a solution iff

$$y_0 \notin \{-\frac{1}{n} \mid n \in \mathbb{N}^*\}$$

Initial Value Encoding FPS

Let $P(x, y, \dot{y}, \dots, y^{(n)})$ be an order n ODE.
To solve P $= 0$, initial values are required but how many?

One encodes an arbitrary long tuple of i.v. by means of a FPS $\bar{y}$.

The "initial value" problem to be solved can then be stated as: given some $\beta \in \mathbb{N}$ such that

$$P(0, \bar{y}) \quad = \quad 0 \mod x^{\beta},$$

does there exist a FPS solution $\bar{\bar{y}}$ prolongating $\bar{y}$:

$$\begin{aligned} \bar{y} &= \bar{\bar{y}} \mod x^{\beta'} \quad \text{and} \\ P(x, \bar{\bar{y}}) &= 0\,? \end{aligned}$$

▸ details on FPS solution computing

Overview

If $\bar{y}$ is a precisely defined FPS then the i.v. problem can be solved (Hurwitz, 1889): from $\bar{y}$, compute β and if

$$P(0, \bar{y}) \quad = \quad 0 \quad \bmod x^{\beta},$$

then the FPS $\bar{\bar{y}}$ exists and can be actually computed.

In [DL84, Thm 3.1] i.v. are provided by polynomial equations and inequations ($\neq 0$) on the coefficients of $\bar{y}$. An algorithm is given to compute a β permitting to decide whether

- there are no FPS $\bar{\bar{y}}$ prolongating any of these $\bar{y}$
- there are FPS $\bar{\bar{y}}$ prolongating some of these $\bar{y}$

Note: [DL84, Thm 3.1] statement is weaker than its proof. But the proof is incomplete. We should have everything needed for a useful software.

Hurwitz (1889) Formula

Let $P(x, y, \dot{y}, \ldots, y^{(n)})$ be an order n ODE.
Let $f_n = \partial P / \partial y^{(n)}$ be the separant of P.
Let $\bar{y}$ be a FPS.
Define k as the valuation of $f_n(x, \bar{y})$ (assuming $f_n(x, \bar{y}) \neq 0$)

$$f_n(x, \bar{y}) \quad = \quad c_0\, x^k + c_1\, x^{k+1} + \cdots \quad (c_0 \neq 0)$$

Let q be a symbol.
There exists a formula for $P^{(2\,k+2+q)}$ with coeffs

f_n	1rst coeff = the separant
$f_{n+1} + q\, f_n'$	2nd coeff, f_n' derivative of the separant
$\vdots$	
$f_{n+r} + \cdots + \binom{q}{r} f_n^{(r)} = A(q)$	let r smallest such that $A(q) \neq 0$
$\vdots$	
$f_{n+k} + \cdots + \binom{q}{k} f_n^{(k)}$	note $f_n^{(k)} = c_0 \neq 0$

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Deduce r and $A(q)$, which is polynomial in q.
The coefficients of $A(q)$ are functions of the coefficients of $\bar{y}$.
Let γ be greater than any integer root of $A(q)$ and define

$$\beta \quad = \quad 2\,k + 2 + \gamma + r\,.$$

If $P(x, \bar{y}) = 0 \mod x^\beta$ then the β first terms of $\bar{y}$ can be prolongated, by a straightforward formula, into a FPS $\bar{\bar{y}}$ such that

$$P(x, \bar{\bar{y}}) \quad = \quad 0\,.$$

The [DL84, Thm 3.1] Idea

The FPS $\bar{y}$ is no more precisely defined: a finite vector t of its coefficients are constrained to lie in some algebraic semi-variety V.

Split the semi-variety so that k and r get uniquely defined. Then $A(t, q)$ is a polynomial in q with coefficients in V.

By universal quantifier elimination, decide whether there exists $\varrho \in \mathbb{N}$ such that, for all $t \in V$, $A(t, \varrho) = 0$.

The existence of such ϱ may prevent FPS solutions to exist: increase the bound β to exceed them and check whether the i.v. system has any solution

$$P(0, \bar{y}) \quad = \quad 0 \quad \text{mod } x^{\beta}\,?$$

This being done or if there does not exist any such ϱ, there are $t \in V$ defining some $\bar{y}$ which can be prolongated. Prolongations need not be unique.

An Example (a "Leaf System" — See Next Slide)

$P = \dot{y}^2 - 3\,y - x^2$

$\bar{y} = y_2\,x^2 + y_3\,x^3 + y_4\,x^4 + \cdots$ (initially V is $y_0 = y_1 = 0$)

Solution: $y(x) = x^2$ or $y(x) = -\frac{1}{4}\,x^2$

$f_n = 2\,\dot{y}$ hence $k = \mathrm{val}(f_n(0,\bar{y})) = 1$.

One finds $r = 1$ and $A(y_2, q) = 4\,y_2\,q + 16\,y_2 - 3$.

Start with $\gamma = 0$.

Thus $\beta = 2\,k + 2 + \gamma + r = 5$.

Update V by equating to 0 the coefficients of $P(0,\bar{y}) \mod x^\beta$ and decompose it into prime components

$$(y_0, y_1, y_2 - 1, y_3, y_4) \cap (y_0, y_1, 4\,y_2 + 1, y_3, y_4)$$

There is no $q \in \mathbb{N}$ such that $A(1, q) = 0$. Thus $\beta_1 = 5$

There is no $q \in \mathbb{N}$ s.t. $A(-\frac{1}{4}, q) = 0$. Thus $\beta_2 = 5$

▸ the algorithm

Return $\beta = \max(\beta_1, \beta_2) = 5$

The Systems Handled by the Algorithm

Systems involve three parts

(1) a finite number of diff. eqns. $P_i = 0$

(2) a finite number of constraints $\mathrm{val}\, H_i \leq k_i$

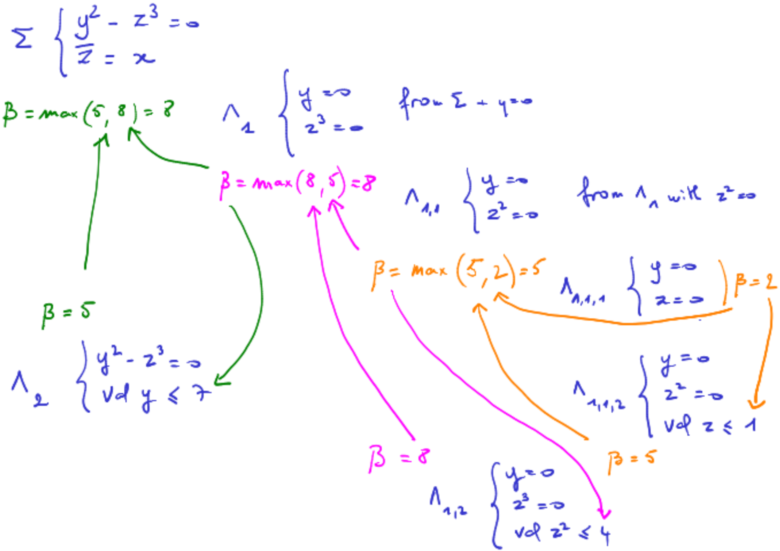
(3) a finite number of polynomial equations and inequations over $\mathbb{Q}$ in the Taylor coefficients of the FPS solutions.

A system is a Leaf System if

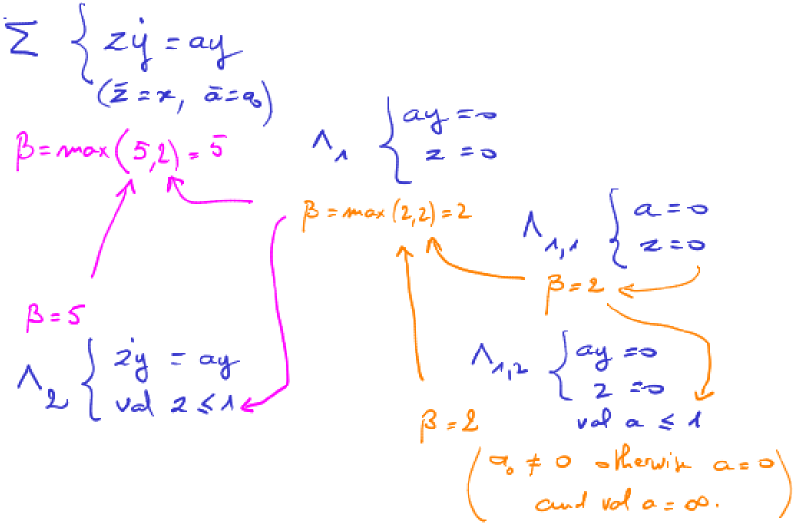
- **(1)** is triangular (in the differential sense)
- the separants of **(1)** occur among the H_i of **(2)**

The algorithm gathers as input a system Σ. It returns a bound β such that Σ has a FPS solution iff it has a solution mod x^β. The FPS solutions are prolongations of the solutions mod x^β. However, the prolongations may not be unique or not exist in some cases.

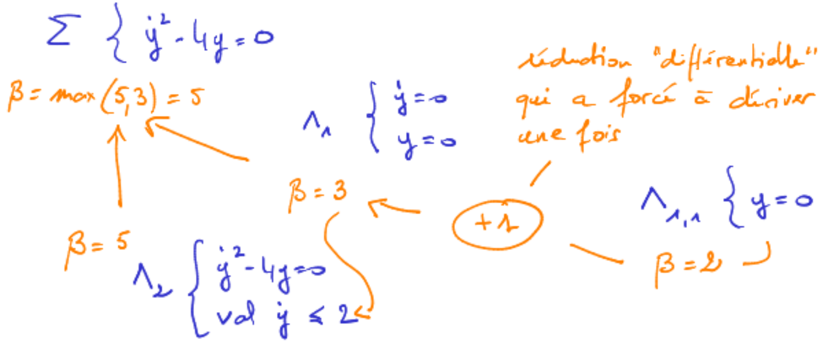
Case Splitting One (Inconsistent System)



Case Splitting Two ($y(x) = 0$ or x^a)



Case Splitting Three (Famous Ritt Example)



The Case Splitting Argument

The algorithm gathers as input a system Σ. It returns a bound β such that Σ has a FPS solution iff it has a solution mod x^β. The FPS solutions are prolongations of the solutions mod x^β. However, the prolongations may not be unique or not exist in some cases.

- Consider an equation $P = 0$.
- Assume inductively that the algorithm works for the extreme case $P = 0,\ f_n = 0$ and returns β_1.
- Then $P = 0$ has FPS solutions if it has solutions mod x^{β_1} which annihilate the β_1 first terms of f_n.
- Among these FPS, the ones such that $\mathrm{val}\, f_n \geq \beta_1$
- The only FPS solutions left (in order to have "iff") are the ones such that $\mathrm{val}\, f_n < \beta_1$.
- Assume inductively this second case works and returns β_2.
- Return $\beta = \max(\beta_1, \beta_2)$.

Handling a Leaf System

The algorithm gathers as input a system Σ. It returns a bound β such that Σ has a FPS solution iff it has a solution mod x^β. The FPS solutions are prolongations of the solutions mod x^β. However, the prolongations may not be unique or not exist in some cases.

- if k, r are not uniquely defined, split cases and return the max. Start with $\gamma = 0$.
- compute $\beta = 2\,k + 2 + \gamma + r$ and $A(q)$ and $\Sigma_1 = \Sigma \mod x^\beta$
- if the ideal (Σ_1) is not prime, split cases
- if the algebraic variety V of Σ_1 is empty return β
- compute $E = \{\varrho \in \mathbb{N} \mid \forall\, t \in V,\ A(t, \varrho) = 0\}$
- if $E = \varnothing$ or $\gamma > \max E$ return β
- take $\gamma = 1 + \max E$ and go back to Step 2

back

Computing Power Series Solutions: Principle

$$\dot{y}^2 + 8\,x\,y - 1 \;=\; 0\,, \quad \text{(unknown function } y(x)\text{)}.$$

Differentiate
$$2\,\dot{y}\,\ddot{y} + 8\,x\,\dot{y} + 8\,y \;=\; 0\,,$$
$$2\,\dot{y}\,y^{(3)} + 2\,\ddot{y}^2 + 8\,x\,\ddot{y} + 16\,\dot{y} \;=\; 0\,,$$
$$\vdots$$

Rename $y^{(i)}$ as y_i. Replace x by the expansion point $\alpha = 0$. Get a system $P_0 = P_1 = \cdots = 0$. Solve it and get some arc

$$(y_0, y_1, y_2, y_3, y_4, y_5, \ldots) \;=\; (0, 1, 0, -8, 0, -64, \ldots)\,.$$

$$\text{Plug in} \quad \bar{\bar{y}} \;=\; \sum \frac{y_i}{i\,!}\,x^i\,.$$

$$\text{Get a FPS solution} \quad \bar{\bar{y}} \;=\; x - \frac{4}{3}x^3 - \frac{8}{15}x^5 + \cdots$$

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$$\text{Plug in} \quad \bar{\bar{y}} \;=\; \sum \frac{y_i}{i\,!}\,x^i\,.$$

$$\text{Argument:} \quad P(0, \bar{\bar{y}}) \;=\; P_0 + P_1\,x + P_2\,\frac{x^2}{2} + \cdots$$

Thus $P(0, \bar{y}) = 0 \mod x^\beta$ requires $P_0 = \cdots = P_{\beta-1} = 0$

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